

AN A_p - A_∞ INEQUALITY FOR THE HILBERT TRANSFORM

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ABSTRACT. We prove in particular that for the Hilbert transform, for $1 < p < \infty$ and a weight $w \in A_p$, that we have the inequality

$$\|H\|_{L^p(w) \rightarrow L^p(w)} \lesssim \|w\|_{A_p}^{1/p} \max\{\|w\|_{A_\infty}^{1/p'}, \|w^{-p'+1}\|_{A_\infty}^{1/p}\}$$

The case of $p = 2$ is an instance of a recent result of Hytönen-Perez, and as a corollary we obtain the well-known bound of S. Petermichl of $\|w\|_{A_p}^{\max\{1, (p-1)^{-1}\}}$. This supports a conjectural inequality valid for all Calderón-Zygmund operators T , and $p \neq 2$.

1. INTRODUCTION: MAIN THEOREM

We are interested in estimates for the norms of Calderón-Zygmund operators on weighted L^p -spaces, a question that has attracted significant interest recently; definitive estimates of this type have been obtained in [H2, HLM⁺], among others. Our particular motivation here is the paper of Lerner [L1], and Hytönen-Perez [HP] that focus on a quantification of good estimates of the norm of an operator in terms of the A_p and A_∞ characteristics of a weight.

1.1. Definition. Let w be a weight on \mathbb{R}^d with density also written as w . Assume $w > 0$ a.e., and $1 < p < \infty$. We define $\sigma = w^{1-p'} = w^{\frac{1}{1-p}}$, which is defined a.e., and set

$$\|w\|_{A_p} := \sup_Q \frac{w(Q)}{|Q|} \left[\frac{\sigma(Q)}{|Q|} \right]^{p-1}.$$

For the endpoint $p = \infty$, we set

$$(1.2) \quad \|w\|_{A_\infty} := \sup_Q w(Q)^{-1} \int_Q M(wQ) \, dx.$$

Note that the definition of (1.2) is different from more familiar definition of an A_∞ norm [H]. It originates in [W], and the article of [HP] makes a convincing case for its central role in the subject. It is known that $\|w\|_{A_\infty} \leq c_d \|w\|_{A_p}$ for $1 < p < \infty$. The article [HP] proves a very sharp estimate on the $L^2(w)$ -norm of Calderón-Zygmund operators in terms of the A_2 and A_∞ characteristics of the weight. *The main purpose of this article is to extend this result to the case*

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of $p \neq 2$ for a few canonical Calderón-Zygmund operators. This result, stated just below, suggests a clear conjecture, which we return to in the concluding section of this paper.

Let us say that T is a classical Calderón-Zygmund operator if it is of the form $Tf(x) = \text{p.v.} \int f(y)K(x-y) dy$ where (1) $K(y) = y^{-1}$, namely the Hilbert transform; (2) $K(z) = z^{-2}$, for complex z , namely the Beurling operator; (3) powers of the Beurling operator; (4) $K(y) = y|y|^{-d-1}$, in dimension $d \geq 2$, namely the Riesz transform; and lastly (4) $K(y)$ is any odd, one-dimensional C^2 Calderón-Zygmund kernel. By the latter, we mean that $|\partial^\epsilon K(y)| \lesssim |y|^{-1-\epsilon}$ for $\epsilon = 0, 1, 2$. These operators are distinguished in that they are known to be in the convex hull of a Haar shift of *bounded* complexity. (See the next section for a definition.)

We define the maximal truncations of T by

$$Tf(x) := \sup_{\epsilon < \delta} \left| \int_{\epsilon < |x-y| < \delta} f(y)K(x-y) dy \right|.$$

This result is new for $p \neq 2$ with or without truncations; and in the case of $p = 2$, it is new for the maximal truncations.

1.3. Theorem. *Let T be a classical singular integral in the sense just defined, and $1 < p < \infty$, and $w \in A_p$. It then holds that*

$$\|T_{\sharp}f\|_{L^p(w)} \leq C_{T,p} \|w\|_{A_p}^{1/p} \max\{\|w\|_{A_\infty}^{1/p'}, \|w^{-p'+1}\|_{A_\infty}^{1/p}\} \|f\|_{L^p(w)}.$$

Since we have $\|w\|_{A_\infty} \lesssim \|w\|_{A_p}$, the result above contains the sharp estimate $\|T_{\sharp}\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\max\{1, 1/(p-1)^{-1}\}}$. This last estimate holds in complete generality, and is a central estimate of [HLM⁺]. There is a weak- L^p norm analog (5.3) of our main inequality that is known in complete generality [HLM⁺, Section 12].

A key step in the proof is the analogous result for dyadic models of the operators in question, see §2. This is a step that has been critical in many contributions, beginning with the breakthrough result of [P2], and has taken an even more central role with the *random* Haar shifts in [H2]. An important component of this definition is a notion of *complexity*.

We apply to the Haar shift operators the remarkable *Lerner median inequality* [L1], see §4. It is well-known that this inequality yields estimates that are exponential in complexity. This means that the main theorem of this section only applies to operators which are in the convex hull of shifts of *bounded complexity*. Fortunately, this is known to include the few canonical examples indicated above. For the Hilbert transform see [P1]; Beurling [DV]; powers of the Beurling [DPV]; Riesz transforms [PTV]; and for C^2 odd one-dimensional kernels, see [V]. It is an open problem to extend the main result of this last paper to higher dimensions.

The Lerner median inequality has been applied in the setting of weighted inequalities, see [CUMP1], but we follow a more refined path here. After application of the Lerner median inequality, we get a class of dyadic positive operators with particular structure. There is a very precise understanding of the norms of these operators, see [LSUT1], and we recall these results in §3, analyzing the particular operators of interest in that section. This analysis completes the proof.

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2. THE HAAR SHIFT RESULT

We begin our dyadic analysis. By a *grid* we mean a collection \mathcal{D} of cubes in \mathbb{R}^d with $Q \cap Q' \in \{\emptyset, Q, Q'\}$ for all $Q, Q' \in \mathcal{D}$. The cubes can be taken to be a product of half-open half-closed intervals. We will say that \mathcal{D} is a *dyadic grid* if each cube $Q \in \mathcal{D}$ these two properties hold. (1) Q is the union of 2^d -subcubes of equal volume (the children of Q), and (2) the set of cubes $\{Q' \in \mathcal{D} : |Q'| = |Q|\}$ partition \mathbb{R}^d . We will work with dyadic grids below.

We give a (standard) definition of *Haar shifts*. By a *Haar function* we will mean a function h_I , supported on I , constant on its children, and orthogonal to χ_I (and no assumption on normalizations). And, by a *generalized Haar function* as a function h_I which is a linear combination of χ_I , and $\{\chi_{I'} : I' \in \text{Child}(I)\}$. Such a function supported on I but need not be orthogonal to constants. In the definition, and throughout the paper, $\ell(Q) = |Q|^{1/d}$ is the side length of Q .

2.1. Definition. For integers $(m, n) \in \mathbb{N}^2$, we say that linear operator \mathbb{S} is a *(generalized) Haar shift operator of parameters (m, n)* if

$$\mathbb{S}f(x) = \sum_{Q \in \mathcal{D}} \sum_{\substack{Q', R' \in \mathcal{D} \\ Q', R' \subset Q}}^{(m, n)} \frac{\langle f, h_{R'}^{Q'} \rangle}{|Q|} h_{Q'}^{R'}$$

where (1) in the second sum, the superscript (m, n) on the sum means that in addition we require $\ell(Q') = 2^{-m}\ell(Q)$ and $\ell(R') = 2^{-n}\ell(Q)$, and (2) the function $h_{R'}^{Q'}$ is a (generalized) Haar function on R' , and $h_{Q'}^{R'}$ is one on Q' , with the joint normalization that

$$(2.2) \quad \|h_{R'}^{Q'}\|_\infty \|h_{Q'}^{R'}\|_\infty \leq 1.$$

In particular, this means that we have the representation

$$\mathbb{S}f(x) = \sum_{Q \in \mathcal{D}} |Q|^{-1} \int_Q f(y) s_Q(x, y) \, dy$$

where $s_Q(x, y)$ is supported on $Q \times Q$, with L^∞ norm at most one. We say that the *complexity* of \mathbb{S} is $\kappa = \max(m, n)$.

Particular examples include Haar multipliers, and the Haar operators central to the result of [P1], in which the Hilbert transform is obtained as a convex combination of a Haar shift of complexity 1. Notice that a Haar shift using only Haar functions is necessarily bounded on L^2 , with norm independent of the complexity type. Dyadic paraproducts are the single example in which one should use generalized Haar functions. The complexity type in this case could be taken to be of type $(0, 0)$, and is explicitly

$$\sum_Q \alpha_Q \mathbb{E}_Q f \cdot h_Q$$

where $|\alpha_Q| \leq \sqrt{|Q|}$, and h_Q is a Haar function. In this case, we further assume that \mathbb{S} is an L^2 -bounded operator. We comment that the analysis of Haar shifts has been central to the papers of [P1, LPR, HPTV, CUMP1] among several other recent publications. Their central role in this paper is expected.

Define the maximal truncations of the Haar shift operator by

$$\mathbb{S}_\sharp f(x) := \sup_{\epsilon > 0} \left| \sum_{Q: \ell(Q) \geq \epsilon} |Q|^{-1} \int_Q f(y) s_I(x, y) \, dy \right|$$

Our analysis extends to the two-weight setting, and we shift to it now. Given a pair of weights w, σ , we set the two weight A_p characteristic to be

$$\llbracket w, \sigma \rrbracket_{A_p} := \sup_Q \left[\frac{w(Q)}{|Q|} \right]^{1/p} \left[\frac{\sigma(Q)}{|Q|} \right]^{1/p'}.$$

To connect this to the classical A_p setting, we would take $\sigma = w^{1-p'}$, where we would then have $\llbracket w, \sigma \rrbracket_{A_p} = \|w\|_{A_p}^{1/p}$. Also, in the norm inequalities in the remainder of the paper, we will have a certain asymmetry between w and σ , one designed so that the inequalities behave well with respect to duality. To dualize, interchange the roles of w and σ , and exchange p for its dual index. This very useful fact appears without comment below.

As we are discussing maximal truncations, the maximal function itself will appear, given by

$$Mf(x) = \sup_{t>0} (2t)^{-d} \int_{[-t,t]^d} |f(x+y)| \, dy$$

The A_p - A_∞ estimate for M have been obtained in [HP]; these considerations are less complicated than those for singular integrals. (One only needs Sawyer's two weight characterization [S1] for the maximal function.)

2.3. Theorem. *For the Maximal function M , index $1 < p < \infty$, and a pair of weights w, σ we have*

$$\|M(f\sigma)\|_{L^p(w)} \leq C_p \llbracket w, \sigma \rrbracket_{A_p} \|\sigma\|_{A_\infty}^{1/p} \|f\|_{L^p(\sigma)}.$$

Observe that the lower bound of $\llbracket w, \sigma \rrbracket_{A_p}$ for the weak-type norm is entirely elementary, thus the estimate above clearly show that contribution of the A_∞ norm to the strong-type norm.

This estimate is equal to or smaller than those we are seeking to prove. Our main technical Theorem is then stated in the two weight language. The corresponding result for the classical operators is of course true.

2.4. Theorem. *For \mathbb{S} an $L^2(\mathbb{R}^d)$ bounded Haar shift operator of complexity κ , index $1 < p < \infty$, and a pair of weights w, σ , it holds that*

$$\|\mathbb{S}_\sharp(\sigma f)\|_{L^p(w)} \leq C_{\mathbb{S},p} \llbracket w, \sigma \rrbracket_{A_p} \left\{ \|w\|_{A_\infty}^{1/p'} + \|\sigma\|_{A_\infty}^{1/p} \right\} \|f\|_{L^p(\sigma)}.$$

Our estimate on the operator norm will be exponential in the complexity parameter κ , so that we will not attempt to quantify it. It is an open question if one can improve this estimate to *polynomial in complexity*, a question we return to in the concluding section of this paper. In particular, a positive answer would immediately imply that our main Theorem holds for *all* continuous Calderón-Zygmund operators.

2.1. Deducing the Main Theorem. If T is a classical singular integral operator, we have the following representation for it. There is a measure space (Ω, \mathcal{A}) , and (not necessarily positive) measure μ on (Ω, \mathcal{A}) , with $0 < |\mu|(\Omega) < \infty$, so that for almost all $\omega \in \Omega$, we have a Haar shift operator \mathbb{S}^ω of complexity at most one, so that for f, g smooth compactly supported functions with $\text{dist}(\text{supp}(f), \text{supp}(g)) > 0$,

$$(2.5) \quad \langle Tf, g \rangle = \langle K * f, g \rangle = \int_{\Omega} \langle \mathbb{S}^\omega f, g \rangle \, d\mu.$$

We remark that if one is interested in the Hilbert transform [P1] or smooth one-dimensional odd Calderón-Zygmund kernels [V], then the measure μ can be taken positive. And indeed, for the Hilbert transform there is a strong form of the equality above [H1]. For the Beurling operator, however, μ will be complex valued [DV]. (And much of the difficulty in that paper is showing that the measure μ is non-trivial!) Thus, our main result, without truncations, follows immediately from the Haar shift version.

Concerning truncations, setting

$$\mathbb{S}^\omega f = \sum_{Q \in \mathcal{D}} |Q|^{-1} \int_I s_Q^\omega(x, y) f(y) \, dy$$

let $\mathbb{S}_\epsilon^\omega$ be the corresponding operator with sum restricted to those q with $\ell(Q) > \epsilon$. Apply (2.5) with f, g so that $\text{dist}(\text{supp}(f), \text{supp}(g)) \geq \epsilon$. We see that the those Q with $\ell(Q) < c\epsilon$ make no contribution in (2.5). From this, it follows that we will have

$$\langle T_\epsilon * f, g \rangle = \int_\Omega \langle \mathbb{S}_\epsilon^\omega f, g \rangle \, d\mu$$

where T_ϵ is an operator with kernel $K_\epsilon(x, y)$ so that for $|x - y| \geq C\epsilon$ we have $K_\epsilon(x, y)$ equals $K(x - y)$, the un-truncated kernel. Moreover, by the normalizing condition (2.2), it follows that

$$|K_\epsilon(x - y)| \lesssim \epsilon^{-d}, \quad |x - y| \lesssim \epsilon.$$

That is, for $|x - y| < C\epsilon$, we have a kernel which at worst performs an average of f . From this, we see that we have

$$T_\epsilon f \lesssim Mf + \int_\Omega \mathbb{S}_\epsilon^\omega f \, d|\mu|.$$

In view of the fact that the maximal function obeys better bounds, see Theorem 2.3, our main theorem for the maximal truncation of classical singular integrals follows from that for Haar shifts.

3. DYADIC POSITIVE OPERATORS

We recall elements of the main results from [LSUT1], which codifies and extends the arguments of [S2, S3]. The latter papers of Sawyer characterized the strong-type two-weight inequalities for the fractional integrals, setting out important elements of the two-weighted theory.

Let $\tau = \{\tau_Q : Q \in \mathcal{D}\}$ be non-negative constants, and define linear operators by

$$T_\tau f := \sum_{Q \in \mathcal{D}} \tau_Q \cdot \mathbb{E}_Q f \cdot \mathbf{1}_Q,$$

Here, we are defining the operator \mathbb{S}_α and a ‘localization’ of \mathbb{S}_α corresponding to a cube R .

Below, we consider the $L^p(\sigma)$ to $L^p(w)$ mapping properties of T_τ , where $1 < p < \infty$. First, we have the weak-type inequalities.

3.1. Theorem. *Let τ be non-negative constants, and w, σ weights. Let $1 < p < \infty$. We have the equivalence below.*

$$\|T_\tau(\sigma \cdot)\|_{L^p(\sigma) \rightarrow L^{p,\infty}(w)} \simeq T_{p'}(w, \sigma) := \sup_{R \in \mathcal{D}} w(R)^{-1/p'} \left\| \sum_{\substack{Q \in \mathcal{D} \\ Q \subset R}} \tau_Q \cdot \mathbb{E}_Q f \cdot \mathbf{1}_Q \right\|_{L^{p'}(\sigma)}.$$

There is a corresponding, harder, strong-type characterization.

3.2. Theorem. *Under the same assumptions as Theorem 3.1 we have the equivalences of norms below.*

$$(3.3) \quad \|T_\tau(\sigma \cdot)\|_{L^p(\sigma) \rightarrow L^p(w)} \simeq T_{p'}(w, \sigma) + T_p(\sigma, w).$$

Notice that the strong type norm is controlled by the larger of two weak-type norms.

3.1. A Particular Class of Operators. We consider a class of operators, motivated by our upcoming application of Lerner's median inequality.

3.4. Definition. We say that a collection \mathcal{L} of dyadic cubes is type L if this condition holds. We have for a constant $\Lambda = \Lambda_{\mathcal{L}} > 0$ so that

$$(3.5) \quad \sup_{Q \in \mathcal{L}} \mathbb{E}_Q \exp \left(\Lambda^{-1} \sum_{Q' \in \mathcal{L} : Q' \subset Q} \mathbf{1}_{Q'} \right) \leq 1.$$

For such a collection \mathcal{L} , define $T_{\mathcal{L}} f := \sum_{Q \in \mathcal{L}} \mathbf{1}_Q \cdot \mathbb{E}_Q f$.

A simple sufficient condition for (3.5) is e.g. that $|\{Q' \in \mathcal{L} : Q' \subsetneq Q\}| < \frac{1}{2}|Q|$. The advantage of the condition above is that it is the minimal condition needed to complete our proof, and it conveniently quantifies our estimate.

The rationale for this definition will be come clear after the discussion in §4. But note that the collection \mathcal{L} is 'thin' in that the \mathcal{L} -children of any cube in the collection must be 'thin.' This notion of thinness depends upon the constant $\Lambda_{\mathcal{L}}$; we will have to take this constant to be exponentially large in the complexity of the Haar shift we consider.

3.6. Proposition. *For a collection of dyadic cubes that is of type L, $1 < p < \infty$, and a pair of weights w, σ , we have*

$$(3.7) \quad \|T_{\mathcal{L}}(\sigma \cdot)\|_{L^p(\sigma) \rightarrow L^{p,\infty}(w)} \lesssim \Lambda[w, \sigma]_{A_p} \|w\|_{A_\infty}^{1/p'},$$

$$(3.8) \quad \|T_{\mathcal{L}}(\sigma \cdot)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim \Lambda[w, \sigma]_{A_p} \max \left\{ \|\sigma\|_{A_\infty}^{1/p}, \|w\|_{A_\infty}^{1/p'} \right\}.$$

We take up the proof of this Proposition. Of the two estimates (3.7) and (3.8), it suffices to prove the weak-type result (3.7). Indeed, it is the content of (3.3), that the strong-type norm of $T_{\mathcal{L}}$ is characterized by the maximum of the two weak-type norms, from $L^p(\sigma)$ to weak- $L^p(w)$ and from $L^{p'}(w)$ to weak- $L^{p'}(\sigma)$. In so doing, we should keep track of the role of the measures and index p . Thus,

$$\begin{aligned} \|T_{\mathcal{L}}(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} &\lesssim \|T_{\mathcal{L}}(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^{p,\infty}(w)} + \|T_{\mathcal{L}}(\cdot w)\|_{L^{p'}(w) \rightarrow L^{p',\infty}(\sigma)} \\ &\lesssim \Lambda\{\llbracket w, \sigma \rrbracket_{A_p} \|\sigma\|_{A_\infty}^{1/p} + \llbracket \sigma, w \rrbracket_{A_p} \|\sigma\|_{A_\infty}^{1/p'}\}. \end{aligned}$$

Then, (3.8) follows as the two-weight A_p terms above are equal.

In particular, it suffices to show that for any cube Q_0 , we have

$$(3.9) \quad \left\| \sum_{\substack{Q \in \mathcal{L} \\ Q \subset Q_0}} \mathbb{E}_Q f w \cdot \mathbf{1}_Q \right\|_{L^{p'}(\sigma)} \lesssim \Lambda \llbracket \sigma, w \rrbracket_{A_p} \|\sigma\|_{A_\infty}^{1/p'} w(Q_0)^{1/p'}.$$

The steps below are the argument pioneered in [LPR], and have been used in [H2, HPTV, HLM⁺, HP]. The details have not been presented before in the positive case, where they are much simpler. We make the definition of the stopping cubes.

3.10. Definition. Let \mathcal{D} be a grid, w a weight. Given a cube $Q \in \mathcal{D}$, we set the *stopping children of Q* , written $\mathcal{T}(Q)$, to be the maximal dyadic cubes $Q' \subset Q$ for which $w(Q')/|Q'| > 4w(Q)/|Q|$. A basic property of this collection is that

$$(3.11) \quad \sum_{Q' \in \mathcal{T}(Q)} |Q'| < \frac{1}{4}|Q|.$$

We set the *stopping cubes of Q_0* to be the collection $\mathcal{S} = \bigcup_{j \geq 0} \mathcal{S}_j(Q_0)$, where we inductively define $\mathcal{S}_0(Q_0) := \{Q_0\}$, and $\mathcal{S}_{j+1}(Q_0) = \bigcup_{Q' \in \mathcal{S}_j(Q_0)} \mathcal{T}(Q')$. Thus, these are the maximal dyadic cubes, so that passing from parent to child in \mathcal{S} , the average value of w is increasing by at least factor 4.

We are free to assume that $Q \subset Q_0$ for all $Q \in \mathcal{L}$, where Q_0 is as in (3.9). Let us fix a non-negative integer a with $2^a \leq 2\llbracket w, \sigma \rrbracket_{A_p}^p$, and integer $b \geq 0$, take $\mathcal{L}_{a,b}(S)$ be those $Q \in \mathcal{L}$ so that (a) $S \in \mathcal{S}(Q_0)$ is the smallest stopping cube which contains Q ; (b) $2^{a-1} \leq \frac{w(Q)}{|Q|} \left[\frac{\sigma(Q)}{|Q|} \right]^{p-1} < 2^a$; and (c) $2^{-b+1}\mathbb{E}_S w \leq \mathbb{E}_Q w \leq 2^{-b+2}\mathbb{E}_S w$. Notice that a holds the A_p ratio fixed, and by definition of the stopping cubes, the sets $\mathcal{L}_{a,b}(S)$ exhaust \mathcal{L} as the three quantities $S \in \mathcal{S}(Q_0)$, and integers a, b vary.

Setting $\mathcal{L}_{a,b} = \bigcup_{S \in \mathcal{S}(Q_0)} \mathcal{L}_{a,b}(S)$, it holds that

$$(3.12) \quad \int_{Q_0} \left| \sum_{\substack{Q \in \mathcal{L}_{a,b} \\ Q \subset Q_0}} \mathbf{1}_Q \mathbb{E}_Q w \right|^{p'} \sigma(dx) \lesssim 2^{-p'b} \|w\|_{A_\infty} 2^{a(p'-1)} w(Q_0).$$

This is summed over $0 \leq a \leq \lceil \log_2 \llbracket w, \sigma \rrbracket_{A_p}^p \rceil + 1$ and non-negative b to prove (3.9).

The essence of the proof of our claim (3.12) is this distributional estimate.

3.13. Lemma. *With the notations above we have these two distributional inequalities, universal over (1) integers a, b, t ; and (2) $S \in \mathcal{S}(Q_0)$ and (3) measure ν equal to Lebesgue measure or σ*

$$(3.14) \quad \nu \left(\left\{ x \in S : \sum_{Q \in \mathcal{L}_{a,b}(S)} \mathbb{E}_Q w \mathbf{1}_Q > K \Lambda 2^{-b} t \mathbb{E}_S w \right\} \right) \lesssim e^{-t} \nu(S).$$

Here, K is a constant.

Proof. Fix data of the Lemma, and note that it suffices to prove that for a maximal $Q_0 \in \mathcal{L}_{a,b}(S)$, that the estimate above holds on the cube Q_0 . Then, (3.14), for ν being Lebesgue measure, obviously reduces to

$$\left| \left\{ x \in Q_0 : \sum_{Q \in \mathcal{L}_{a,b}(S)} \mathbf{1}_Q > K \Lambda t \right\} \right| \lesssim e^{-t} |Q_0|$$

which is an immediate consequence of \mathcal{L} being of type L, see (3.5). So, we consider the case of ν being σ . But note that with $t > 1$ fixed, the event

$$\left\{ x \in Q_0 : \sum_{Q \in \mathcal{L}_{a,b}(S)} \mathbb{E}_Q w \mathbf{1}_Q > K \Lambda t 2^{-b} \mathbb{E}_S w \right\}$$

is a union of disjoint cubes in collection $\mathcal{E}_t \subset \mathcal{L}_{a,b}$. Now, for each $Q \in \mathcal{E}_t$, we have held the A_p characteristic essentially constant, namely about 2^a . And $\mathbb{E}_Q w$ is held to be approximately $2^{-b} \mathbb{E}_S w$. Thus, $\sigma(Q)$ is, up to a fixed multiple, a constant times $|Q|$. Call this multiple ρ , which is a function of the stopping cube S , but the latter is held fixed. Hence,

$$\begin{aligned} \sigma \left\{ x \in S : \sum_{Q \in \mathcal{L}_{a,b}(S)} \mathbb{E}_Q w \mathbf{1}_Q > K t 2^{-b} \mathbb{E}_S w \right\} &= \sum_{Q \in \mathcal{E}_t} \sigma(Q) \\ &\simeq \rho \sum_{Q \in \mathcal{E}_t} |Q| \lesssim \rho e^{-t} |Q_0| \simeq e^{-t} \sigma(Q_0), \end{aligned}$$

Our proof is complete. \square

We can now verify (3.12). We will write

$$\sum_{Q \in \mathcal{L}_{a,b}} \mathbb{E}_Q w \cdot \mathbf{1}_Q \lesssim \sum_{S \in \mathcal{S}(Q_0)} 2^{-b} \mathbb{E}_S w \mathbf{U}_S$$

where $\mathbf{U}_S := \sum_{Q \in \mathcal{L}_{a,b}(S)} \mathbf{1}_Q$. There is one more variable that is useful for us to hold essentially constant. Define an event $E_{S,0} = \{\mathbf{U}_S < \Lambda 2^{n+1}\}$, and for $n > 0$, set $E_{S,n} = \{2^n \Lambda \leq \mathbf{U}_S < 2^{n+1} \Lambda\}$. Then, we have $\sigma(E_{S,n}) \lesssim e^{-c2^n} \sigma(S)$ for $c = c_{\mathcal{L}} > 0$. Moreover, we can estimate, using a familiar trick,

$$\begin{aligned} \left[\sum_{S \in \mathcal{S}(Q_0)} \mathbb{E}_S w \cdot \mathbf{U}_S \right]^{p'} &= \left[\sum_{n=0}^{\infty} \sum_{S \in \mathcal{S}(Q_0)} 2^{-b-n/p'+n/p'} \mathbb{E}_S w \cdot \mathbf{U}_S \mathbf{1}_{E_{S,n}} \right]^{p'} \\ &\lesssim \sum_{n=0}^{\infty} 2^n \left[\sum_{S \in \mathcal{S}(Q_0)} 2^{-b} \mathbb{E}_S w \cdot \mathbf{U}_S \mathbf{1}_{E_{S,n}} \right]^{p'} \\ &\lesssim \sum_{n=0}^{\infty} 2^n \sum_{S \in \mathcal{S}(Q_0)} \left| 2^{-b} \mathbb{E}_S w \cdot \mathbf{U}_S \mathbf{1}_{E_{S,n}} \right|^{p'}. \end{aligned}$$

The first line follows by an appropriate choice of Hölder's inequality, and the second as the sum, for each point $x \in Q_0$ is a super-geometric series of numbers. This is wasteful in n , but by (3.14),

$$\begin{aligned} 2^n \int_{E_{S,n}} \left| 2^{-b} \mathbb{E}_S w \mathbf{U}_S \right|^{p'} \sigma(dx) &\lesssim \Lambda^{p'} 2^{-bp'-n} [\mathbb{E}_S w]^{p'} \sigma(S) \\ &\lesssim L^{p'} 2^{-bp'+a(p'-1)-n} w(S). \end{aligned}$$

The last line follows by trading out the A_p characteristic, which is approximately 2^a on these stopping cubes. We can of course trivially sum this in $n \geq 0$. Now, the A_{∞} property is decisive. We employ the elementary property (3.11) to estimate

$$\begin{aligned} \sum_{S \in \mathcal{S}(Q_0)} w(S) &= \int_{Q_0} \sum_{S \in \mathcal{S}(Q_0)} \frac{w(S)}{|S|} \mathbf{1}_S dx \\ &\lesssim \int_{Q_0} M(w \mathbf{1}_{Q_0}) dx \leq \|w\|_{A_{\infty}} w(Q_0). \end{aligned}$$

Combining estimates, we have proved (3.12).

4. LERNER'S MEDIAN INEQUALITY; APPLICATION TO HAAR SHIFT OPERATORS

We recall definitions for the inequality from [L1], which applies to a measurable function ϕ on \mathbb{R}^d , and cube Q . A median of ϕ restricted to Q , is a possibly non-unique real number such that

$$\max\{|\{x \in Q : \phi(x) > m_\phi(Q)\}|, \max\{|\{x \in Q : \phi(x) > m_\phi(Q)\}|\}\} \leq \frac{1}{2}|Q|.$$

For parameter $0 < \lambda < 1$, we define an measure of oscillation of f to be

$$\omega_\lambda(\phi; Q) := \inf_{c \in \mathbb{R}} ((\phi - c)\mathbf{1}_Q)^*(\lambda|Q|).$$

Here, ϕ^* denotes the non-increasing rearrangement, so that if ϕ is supported on Q , $\phi^*(\lambda|Q|)$ is the λ^{th} percentile of ϕ . The local sharp maximal function of f is

$$M_{\lambda;Q}^\# \phi(x) := \sup_{Q' \subset Q} \mathbf{1}_{Q'} \omega_\lambda(\phi, Q').$$

4.1. Theorem (Lerner). *Let ϕ be a measurable function on \mathbb{R}^d , and Q_0 a cube. Then, there is a collection of cubes $\{Q_j^\ell\}$ all dyadic cubes contained in Q_0 so that*

(1) *We have the pointwise inequality*

$$|\phi(x) - m_\phi(Q_0)| \lesssim M_{1/4,Q_0}^\# \phi(x) + \sum_{\ell=1}^{\infty} \sum_j \omega_{2^{-d-2}}(f, \hat{Q}_j^\ell) \mathbf{1}_{Q_j^\ell}(x)$$

where \hat{Q} is the parent of dyadic cube Q .

(2) *The cubes Q_j^ℓ are disjoint in j , with k fixed.*

(3) *Setting $\Omega_k = \bigcup_j Q_j^\ell$, we have $\Omega_{k+1} \subset \Omega_k$,*

(4) *$|Q_j^\ell \cap \Omega_{k+1}| < \frac{1}{2}|Q_j^\ell|$.*

The collection $\mathcal{L} = \{Q_j^\ell : j, \ell \geq 1\}$ is a collection of dyadic intervals of type L, with $\Lambda_{\mathcal{L}} \simeq 1$. We will say that $\mathcal{M} \subset \mathcal{L}$ has generations separated by t if it is a subset of

$$\{Q_j^\ell : j \geq 1, \ell \equiv t' \pmod{t}, \quad 0 \leq t' < t.$$

We turn to the application of this inequality to Haar shift operators \mathbb{S}_\sharp . We will show that we can dominate \mathbb{S}_\sharp by a sum of dyadic positive operators of type L. The sum has a number of terms in in controlled by complexity. Then our technical Theorem 2.4 follows from Proposition 3.6. In order to control the measure of oscillations above, as it now standard in the subject, we appeal to a weak- L^1 estimate.

4.2. Lemma. *Let \mathbb{S} be an L^2 bounded Haar shift operator of complexity κ . We then have $\|\mathbb{S}_\sharp f\|_{1,\infty} \lesssim \kappa \|f\|_1$.*

Aside from the linear bound in complexity, this is a standard argument; details can be found in [H2, Proposition 5]. We need to make these comments on the estimation of the oscillation terms above for Haar shift operators. Recall that \mathbb{S} is a Haar shift operator of complexity κ . Fixing cube Q , and letting $Q^{(\kappa)}$ be the κ -fold parent of Q , it follows that if measure g is *not supported on* $Q^{(\kappa)}$, that the function $\mathbb{S}_\sharp(g)$ is constant on Q . (Note that this is certainly not true for continuous Calderón-Zygmund operators, so this proof seems to be limited to the dyadic setting.) Constants do not contribute to the measure of oscillation that we are concerned with, therefore, in estimating $\omega_{2^{-d-2}}(\mathbb{S}_\sharp(f\sigma), Q)$, we can assume that f is supported on $Q^{(\kappa)}$. Moreover, in seeking to estimate this oscillatory term, we can group all the scales inside Q , and appeal to the weak-type estimate. For the κ -scales above Q , we use the size condition (2.2). From this, we see that

$$\left| \left\{ x \in Q : \mathbb{S}_\sharp(f \mathbf{1}_{(Q)^{(\kappa)}} \sigma) \geq K \kappa \mathbb{E}_Q |f| \sigma + \sum_{t=1}^{\kappa+1} \mathbb{E}_{(Q)^{(t)}} |f| \sigma \right\} \right| \leq 2^{-d-2} |Q|$$

for a dimensional constant K . Hence, we conclude that

$$\omega_{2^{-d-2}}(\mathbb{S}_\sharp(f\sigma), \widehat{Q}_j^\ell) \lesssim \kappa \mathbb{E}_Q |f| \sigma + \sum_{t=1}^{\kappa} \mathbb{E}_{(Q)^{(t)}} |f| \sigma.$$

From this, it follows that we have the following estimate on the local sharp maximal function, $M_{\lambda, Q}^\sharp(\mathbb{S}_\sharp f) \lesssim \kappa M |f| \sigma$.

Now, let f be supported on a fixed dyadic cube Q_1 . We apply Theorem 4.1 to $\mathbb{S}_\sharp(f\sigma)$, restricted to a cube Q_0 , much larger than Q_1 . We can estimate

$$(4.3) \quad \left| \mathbb{S}_\sharp(f\sigma) - \omega_{1/2}(\mathbb{S}_\sharp(f\sigma), Q_0) \right| \lesssim M_{\lambda, Q}^\sharp(\mathbb{S}_\sharp f) + \sum_{\ell=1}^{\infty} \sum_j \omega_{2^{-d-2}}(\mathbb{S}_\sharp(f\sigma), \widehat{Q}_j^\ell) \mathbf{1}_{Q_j^\ell}$$

We have already seen that the local sharp function is bounded by $\kappa M(f\sigma)$. The structure of the Haar shift operator shows that

$$\omega_{2^{-d-2}}(\mathbb{S}_\sharp(f\sigma), \widehat{Q}_j^\ell) \lesssim \kappa \mathbb{E}_{\widehat{Q}_j^\ell} \sigma |f| + \sum_{t=2}^{\kappa} \mathbb{E}_{(Q_j^\ell)^{(t)}} \sigma |f|$$

We have again used Lemma 4.2, for the first term on the right.

We will show that

$$(4.4) \quad \left\| \sum_{\ell=1}^{\infty} \sum_j \mathbb{E}_{(Q_j^\ell)^{(t)}} \sigma |f| \cdot \mathbf{1}_{Q_j^\ell} \right\|_{L^p(w)} \lesssim C_{p,t} \llbracket w, \sigma \rrbracket_{A_p} \left\{ \|w\|_{A_\infty}^{1/p'} + \|\sigma\|_{A_\infty}^{1/p} \right\} \|f\|_{L^p(\sigma)}.$$

This shows that the right hand side of (4.3) is bounded with a norm estimate that depends upon complexity. (It will be exponential.) Assuming that f is compactly supported, and taking Q_0 arbitrarily large, we can make $m_\phi(Q_0)$ as small as we wish. So by Fatou Theorem, we will have finished the proof.

The maximal function obeys our estimate, see Theorem 2.3, bringing our focus to the remaining terms in (4.3). The main point is this: The term in (4.4) is dominated by an operators of type L, with constant $\Lambda_L \lesssim 2^{td}$. From this, and (3.8), the required estimate (4.4) follows immediately.

Fix $1 \leq t \leq \kappa + 1$, and note the following. Fix a cube R , and consider \mathcal{R} , the collection of those $Q_j^\ell \in \mathcal{L}$ such that $(Q_j^\ell)^{(t)} = R$. The collection \mathcal{R} consists of disjoint cubes. This means that below, we can work with a set of indices $(\ell, j) \in \mathbb{K}$ with the defining property of \mathbb{K} being that for all pairs of integers $Q_j^\ell \in \mathcal{M}$ there is a unique $(\ell', j') \in \mathbb{K}$ with $(Q_j^\ell)^{(t)} = (Q_{j'}^{\ell'})^{(t)}$. We argue that this operator is of type L with constant $\Lambda_L \lesssim 2^{td}$.

$$\sum_{(\ell, j) \in \mathbb{K}} \mathbb{E}_{(Q_j^\ell)^{(t)}} |f| \sigma \cdot \mathbf{1}_{(Q_j^\ell)^{(t)}}$$

Indeed, for any cube R , we have

$$\left\| \sum_{\substack{(\ell, j) \in \mathbb{K} \\ (Q_j^\ell)^{(t)} \subset R}} \mathbf{1}_{(Q_j^\ell)^{(t)}} \right\|_1 \leq 2^{td} \left\| \sum_{\substack{(\ell, j) \in \mathbb{K} \\ Q_j^\ell \subset R}} \mathbf{1}_{Q_j^\ell} \right\|_1 \lesssim 2^{td} |R|$$

by property (4) of Theorem 4.1. This estimate is uniform in R , hence, by a well-known John-Nirenberg argument, it shows that $\Lambda_L \lesssim 2^{td}$, completing our proof.

4.5. *Remark.* Our approach gives a proof of the weak-type estimate (5.3) from Lerner's inequality, one of the few inequalities missing from the papers [CUMP2, CUMP1]. One should note that the weak-type inequality for the dyadic positive operators is quite easy. See [LSUT1, S2].

5. CONCLUDING REMARKS

Our motivation for writing this paper is to present positive evidence for this conjecture.

5.1. **Conjecture.** For T an $L^2(\mathbb{R}^d)$ bounded Calderón-Zygmund Operator and $1 < p < \infty$, and $w \in A_p$ it holds that

$$\|T_{\sharp} f\|_{L^p(w)} \leq C_{T,p} \|w\|_{A_p}^{1/p} \max\{\|w\|_{A_\infty}^{1/p'}, \|w^{-p'+1}\|_{A_\infty}^{1/p}\} \|f\|_{L^p(w)}.$$

Currently, this is known for the un-truncated operator and $p = 2$, [HP]. For the definition of a Calderón-Zygmund operator, we refer the reader to *op. cit.* An attractive part of this conjecture,

pointed out to the author by Tuomas Hytönen, is that for canonical examples of T , it is a standard part of the subject to have a lower bound on $\|T\|_{L^p(w) \rightarrow L^p(w)}$ of $\|w\|_{A_p}^{1/p}$. Thus, the form of the estimate above quantifies the A_∞ contribution to the norm.

The corresponding weak-type result is contained in [HLM⁺, Theorem 12.3], recalled here as it seems to be the strongest known estimates, in the case of $p \neq 2$.

5.2. Theorem. *For T an $L^2(\mathbb{R}^d)$ bounded Calderón-Zygmund Operator and $1 < p < \infty$,*

$$(5.3) \quad \|T_b f\|_{L^{p,\infty}(w)} \leq C_{T,p} \|w\|_{A_p}^{1/p} \|w\|_{A_\infty}^{1/p'} \|f\|_{L^p(w)},$$

$$(5.4) \quad \|T_b f\|_{L^p(w)} \leq C_{T,p} \left(\|w\|_{A_p}^{1/p} \|w\|_{A_\infty}^{1/p'} + \|w\|_{A_p}^{1/(p-1)} \right) \|f\|_{L^p(w)}.$$

It is a curious remark that the extrapolation estimates in [HP] do not give better than the estimate above, despite extrapolating from the sharp L^2 estimate. Likewise, our Main Theorem cannot be proved by the same elementary arguments from [CUMP1].

What is required to give a proof of the conjecture? In the singular integral case, we do not have a principle matching that of Sawyer's observation in the positive operator case that the strong type norm is dominated by the maximum of two weak type norms. Instead, the best results in the two weight case are contained in [LSUT2, LSUT3, HLM⁺]. The cleanest and simplest argument in [HLM⁺, Theorem 4.7], which proves a general two-weight estimate for Haar shift operators that is linear in complexity. It is largely satisfying, except for the presence of the 'non-standard' testing condition (4.8). Indeed, the contribution of the non-standard testing condition to the estimate in (5.4) is the term $\|w\|_{A_p}^{1/(p-1)}$ by which we miss the conjecture. Any essential strengthening of this Theorem would be interesting, and a potential step towards proving the conjecture above.

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